Theory of sound amplification by stimulated emission of radiation with consideration for coagulation

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The theoretical scheme of sound amplification by stimulated emission of radiation (SASER) is considered. Liquid with gas bubbles serves as the active medium. The pumping is produced by an alternating electric field. Phase bunching is realized by acoustic radiation forces. The influence of a coagulation of the bubbles on the SASER operation is considered. Generation conditions are found analytically. The nonlinear stage of SASER operation and the saturation mode are investigated by numerical methods. The emission of a SASER in steady state and the direction pattern for a SASER are considered. [S1063-651X(97)03607-6]

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I. INTRODUCTION

At the present time there are many types of lasers—devices generating coherent electromagnetic waves at the sacrifice of the stimulated emission (or scattering) of light by an active medium. The existing lasers cover a wide wavelength range—from ultraviolet to submillimeter. However, to our knowledge acoustic analogs of such devices have not been created up to now despite the progress in laser technology.

Meanwhile, a problem of creation of an ''acoustic laser'' (which will be referred below as a SASER—sound amplification by stimulated emission of radiation) is of great interest because of a variety of potential applications of such devices. As generators of shock waves, they can be used for impact action on underwater and air objects, in medical applications (such as the destruction of stones in the kidneys), etc.

Recently a theoretical scheme of a SASER with electric pumping was proposed in a number of papers by the present authors [1–5]. A liquid dielectric with uniformly distributed dispersed particles was suggested as an active medium. Different types of oils as well as routine distilled water can be used as a liquid dielectric. Gas bubbles resulting from electrolysis can serve as dispersed particles. Pumping is excited in an active medium, leading to periodic pulsations of dispersed particle volumes. The initial distribution of the particles is spatially homogeneous. As a result, the waves emitted by particles are added to the different phases. Consequently, the resulting pressure of the useful wave is equal to zero.

However, for the active medium in the resonator, an acoustic mode can be excited. Then the particles can be bunched due to acoustic radiation forces. This leads to self-synchronization of the oscillating particles, and to the amplification of the useful mode. The suggested scheme of the SASER is analogous to the free-electron laser (FEL) [6]. In FEL theory the electromagnetic emission is created by electron beams moving through magnetic periodic systems. These systems are called undulators or wigglers. Undulators play the role of pumping. Initially the emission of electrons is not coherent, but the electrons become grouped due to their interaction with the applied electromagnetic wave. As a result, the emission becomes coherent. This leads to the amplification of the electromagnetic field.

It should be noted that alternative theoretical schemes for SASER's were suggested in a number of earlier papers. For instance, mode self-synchronization and acoustic field amplification by a gas-liquid mixture were theoretically considered by Kobelev, Ostrovsky, and Soustova *et al.* [7] using a model of incoherent mechanical monopole oscillators for the gas fraction. Sound oscillations in the Helmholtz resonator filled with a supercooled vapor were investigated by Kotusov and Nemtsov [8]. The energy liberation in that system results from the fact that condensation proceeds more actively than evaporation. A part of this energy is spent on the amplification of sound. However, the schemes for SASER's suggested in these papers have not been realized in practice due to weak self-synchronization.

In the case of ordinary piezoelectric emitters which are usually used in the generation of ultrasound, only a working surface emits and, consequently, such devices are two-dimensional system, systems. The SASER differs from the above system particularly in that it is a three-dimensional system, because the whole volume of an active medium is emitted.

The scheme of a SASER with pumping by an electric field, as well as the generation conditions, were investigated by Zavtrak [1,2]. It was shown that two types of losses must be overcome. The first type is caused by energy dissipation in the active medium, and the second by radiation losses on the faces of the resonator. However, in the case of electric pumping the amplitude of the electric field (and hence pumping pressure) is limited by the value of the electric puncture intensity of liquid dielectrics. This limitation can be prevented by the use of mechanical pulsations of the resonator walls. The generation conditions for SASER's with plane and cylindrical resonators and mechanical pumping were evaluated [3–5]. Analysis of the impact of the continuous distribution of bubbles by radii on the SASER operation [5] showed that in this case bubbles with a resonance cyclic frequency are of crucial importance in the amplification of the applied mode.

In the above-mentioned papers it was assumed that the deviation of the spatial concentration of bubbles and useful waves were less than the initial concentration and pumping,

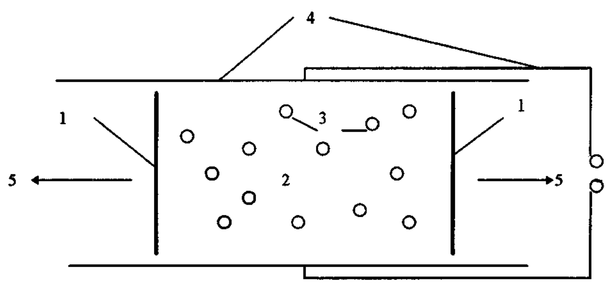

FIG. 1. Scheme of the SASER with electric pumping. (1) Walls of sound resonator. (2) Active medium (liquid dielectric with gas bubbles. (3) Gas bubbles. (4) Electromagnetic system which creates the periodic electric field. (5) Sound emission.

respectively. In the present paper we argue against this assumption, considering SASER operation for all time. In addition, to obtain more realistic estimations, in this paper we shall analyze the role of coagulation of bubbles. To simplify the calculations, we restrict our consideration to the case of a SASER with electric pumping and equal bubbles.

II. BASIC FORMULA

The scheme of a SASER with electric pumping is shown in Fig. 1. The active medium is placed between two planes. The radiation propagates along the z axis, and L is the length of the resonator in this direction.

The dynamics of the gas-liquid mixture is described by the following set of equations [9]:

$$\Delta P' - \frac{1}{c_l^2}\frac{\partial^2 P'}{\partial t^2} - (\alpha + i\beta)P' = (\alpha + i\beta)P_E \exp(i\omega t), \tag{1}$$

$$\xi \mathbf{U} = (\mathrm{Re}A)\nabla |P|^2 - i(\mathrm{Im}A)(P^*\nabla P - P\nabla P^*), \tag{2}$$

$$\frac{\partial n}{\partial t} + \mathrm{div}(n\mathbf{U}) = I_1 + I_2 + I_3. \tag{3}$$

The first equation describes the sound pressure wave propagating in the active medium. Here, c_l is the velocity of sound in the pure liquid; P_E and ω are the amplitude and frequency of the pumping, respectively; and P' is the useful pressure wave. The quantities α and β are described by following expressions:

$$\alpha = \alpha(\mathbf{r}, R_0, t) = -4\pi \int_0^\infty (\mathrm{Re}A) n(\mathbf{r}, R_0, t)\, dR_0, \tag{4}$$

$$\beta = \beta(\mathbf{r}, R_0, t) = -4\pi \int_0^\infty (\mathrm{Im}A) n(\mathbf{r}, R_0, t)\, dR_0, \tag{5}$$

where

$$A = \frac{R_0}{(\omega_0^2/\omega^2) - 1 + i\delta} \tag{6}$$

is the scattering amplitude of sound on a bubble, ω_0 is the proper frequency of the bubble with radius R_0, and δ is the absorption constant. The quantity $n(\mathbf{r}, R_0, t)$ is the distribution function of the particles by radii (n is equal to the number of bubbles with mean radii from R_0 to $R_0 + dR_0$ in a unit of volume in the vicinity of point $\mathbf{r}$). The initial concentration of the bubbles in liquid is spatially homogeneous ($n = n_0$, $\alpha = \alpha_0$, and $\beta = \beta_0$).

Equation (2) describes the translational motion of a bubble. Here $\mathbf{U}$ is the translational velocity of a bubble, $P = P_E \exp(i\omega t) + P'$ is the resulting pressure (static pressure is skipped) acting on the bubbles, and $\xi = 4\rho_l \mu_l \omega^2 R_0$, where μ_l and ρ_l are the viscosity and density of a pure liquid, respectively.

Finally, Eq. (3) is the balance equation of the number of particles in a unit of the phase volume $d^3\mathbf{r}dR_0$. The right-hand side of Eq. (3) represents the integral of collisions, which involves the spectrum changes of bubbles under coagulation; I_1, described by the arrival of the bubbles from the source (it might be an electrolysis, for instance); the quantity

$$I_2 = -n(R_0) \int_0^\infty n(R_0') \sigma(R_0, R_0') |\mathbf{U}(R_0) - \mathbf{U}(R_0')| dR_0', \tag{7}$$

gained by the departure of the bubbles from the environs of R_0; and the quantity

$$I_3 = \frac{1}{2}\int_0^{R_0} n(R_0'') n(R_0') \sigma(R_0'', R_0') |\mathbf{U}(R_0'') - \mathbf{U}(R_0')| \frac{R_0^2}{(R_0'')^2} dR_0' \tag{8}$$

described by increasing the number of the bubbles formed from the bubbles of smaller size. The radii R_0' and R_0'' are determined from the law of conservation mass of a gas in the bubble. In Eqs. (7) and (8), σ is the cross section of collisions of two bubbles with mean radii R_1 and R_2 determined by the following expression:

$$\sigma |\mathbf{U}(R_1) - \mathbf{U}(R_2)| = \begin{cases} 4\pi\kappa, & \kappa \geqslant 0 \\ 0, & \kappa < 0, \end{cases} \tag{9}$$

where

$$\kappa = \frac{(R_1 + R_2)\left[\left(\frac{\omega_1^2}{\omega^2} - 1\right)\left(\frac{\omega_2^2}{\omega^2} - 1\right) + \delta_1\delta_2\right]}{3\mu_l\rho_l\omega^2\left[\left(\frac{\omega_1^2}{w^2} - 1\right)^2 + \delta_1^2\right]\left[\left(\frac{\omega_2^2}{w^2} - 1\right)^2 + \delta_2^2\right]} |P|^2, \tag{10}$$

and ω_i and δ_i ($i = 1$ and 2) are the proper frequency and absorption constants of the bubbles, respectively.

In view of complexity to analyze set (1)–(3), we accept some simplifying assumptions; that is, we will suppose that all bubbles have an equal radius R_0 greater than the proper radius corresponding to the pumping frequency ω. Then one can neglect the influence of the merged bubbles on a sound field. At that $I_3 = 0$ and expressions (4), (5), and (7) are easily calculated. Finally, we will consider only one-dimensional solutions of Eqs. (1)–(3).

Let us represent pressure P' as $P'(t, z) = \overline{P(t)} \exp(i\omega t)$. Applying Bogolubov-Mitropol'sky's averaging method [10] to Eq. (1) over the high-frequency component $\exp(i\omega t)$, and substituting Eq. (2) into (3), one can obtain

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\omega^2}{c_l^2}-\frac{2i\omega}{c_l^2}\frac{\partial}{\partial t}\right)\overline{P}=-4\pi An(\overline{P}+P_E), \quad (11a)$$

$$\frac{\partial n}{\partial t}+\frac{1}{\xi}\frac{\partial}{\partial z}\left(n\left[A^*(P_E+\overline{P^*})\frac{\partial\overline{P}}{\partial z}+A(P_E+\overline{P})\frac{\partial\overline{P^*}}{\partial z}\right]\right)$$
$$=I_1-\frac{8\pi R_0 n^2|P_E+\overline{P}|^2}{3\mu_l\rho_l\omega^2\left[\left(\frac{\omega_0^2}{\omega^2}-1\right)^2+\delta^2\right]}. \quad (11b)$$

III. ANALYSIS OF EQUATIONS IN THE INITIAL STAGE OF GENERATION

In the initial stage of generation (at small t), we will seek the useful mode in the form of a standing wave,

$$\overline{P(z,t)}=P_0(t)\cos k_L z, \quad (12)$$

with $k_L=\pi m/L, m=1,2, \ldots$. Let $n'(\mathbf{r},t)$ be the deviation from the initial distribution function n_0, i.e.,

$$n(\mathbf{r},t)=n_0+n'(\mathbf{r},t), \quad (13)$$

and $|n'|\ll n_0$, and $|P_0|\ll|P_E|$ [1,2]. Substituting Eqs. (12) and (13) into Eqs. (11a) and (11b), and linearizing of these equations over n' and P_0, gives the following:

$$\left(\frac{\omega^2}{c_l^2}-k_L^2+4\pi An_0-\frac{2i\omega}{c_l^2}\frac{d}{dt}\right)P_0\cos k_L z$$
$$=-4\pi AP_E(n_0+n'), \quad (14)$$

$$\frac{\partial n'}{\partial t}=\frac{k_L^2 n_0 P_E}{\xi}(A^*P_0+AP_0^*)\cos k_L z+I_1$$
$$+\gamma(n_0P_E+2P_En'+n_0(P_0+P_0^*)\cos k_L z) \quad (15)$$

where

$$\gamma=-\frac{8\pi R_0 n_0 P_E}{3\mu_l\rho_l\omega^2\left[\left(\frac{\omega_0^2}{\omega^2}-1\right)^2+\delta^2\right]}.$$

The solution of Eq. (15) has the form

$$n'(z,t)=\frac{I_1+\gamma n_0 P_E}{2P_E\gamma}[\exp(2P_E\gamma t)-1]+n_2(t)\cos k_L z. \quad (16)$$

The first item in the right-hand side of this expression describes the change of the total number of bubbles in a unit of volume. This takes place due to the coagulation and source. The latter represents the spatial redistribution of bubbles under acoustic radiation forces. The quantity n_2 can be found from the equation as follows:

$$\frac{dn_2}{dt}=\frac{k_L^2 n_0 P_E}{\xi}(A^*P_0+AP_0^*)+\gamma n_0(P_0+P_0^*). \quad (17)$$

After multiplying Eqs. (14) and (15) by $\cos k_L z$ and averaging them over z, we reduce this set to the wave equation

$$\left(\frac{\omega^2}{c_l^2}-k_L^2+4\pi An_0-\frac{2i\omega}{c_l^2}\frac{d}{dt}\right)\frac{dP_0}{dt}$$
$$=-\frac{4\pi An_0k_L^2P_E^2}{\xi}(A^*P_0+AP_0^*)$$
$$+2\gamma P_E\left(\frac{\omega^2}{c_l^2}-k_L^2+4\pi An_0-\frac{2i\omega}{c_l^2}\frac{d}{dt}\right)P_0$$
$$-4\pi AP_E\gamma n_0(P_0+P_0^*). \quad (18)$$

In deriving this equation, it was taken into account that the liquid gas content is small, i.e. $|\alpha_0+i\beta_0|\ll\omega^2/c_l^2$.

Let us represent the solution of Eq. (18) as

$$P_0(t)=A^+\exp(i\omega' t)+A^-\exp(-i\omega'^* t),$$

where ω' is a complex quantity. Then, with the proviso that $|A^-|\ll|A^+|$ [2], one finds that

$$\mathrm{Re}\,\omega'\approx-\frac{c_l^2}{2\omega}\left(\frac{\omega^2}{c_l^2}-k_L^2+4\pi n_0\mathrm{Re}A\right)\equiv-\frac{\delta_L\omega}{2},$$

$$\mathrm{Im}\,\omega'\approx-\frac{c_l^2}{2\omega}\left\{\frac{2}{\delta_L\omega}\left[\frac{4\pi|A|^2P_E^2k_L^2n_0}{\xi}\right.\right.$$
$$\left.-2\gamma P_E\left(\frac{\omega^2}{c_l^2}-k_L^2+2\pi n_0\mathrm{Re}A\right)\right]+4\pi n_0\mathrm{Im}A$$
$$\left.+\frac{4\gamma\omega P_E}{c_l^2}\right\}. \quad (19)$$

The real part of ω' defines the frequency shift between the pumping frequency ω and the proper frequency of the resonator $\omega_L=c_l\sqrt{k_L^2+\alpha_0}$; the image part defines the increment of the useful wave amplification. One can seen from Eq. (19) that generation takes place when $\delta_L>0$, and

$$P_E\geqslant P_{\mathrm{st}}\approx\frac{1}{\left(\frac{k_L^2}{\delta_L\omega}+\frac{32\omega}{c_l^2}\right)^{1/2}}\sqrt{2\mu_l\rho_l\delta\omega^2}\approx\frac{1}{k_L}\sqrt{2\mu_l\rho_l\delta\delta_L\omega^3}. \quad (20)$$

The quantity P_{st} defines the starting (threshold) pumping pressure for the beginning of generation [1,2]. The right-hand side of Eq. (20) involves only the member associated with the sound energy dissipation in the active medium, but not the member attributed to the radiation losses on the faces of the resonator. This is due to the fact that, for simplicity, the useful wave was chosen in the form of a standing wave corresponding to the absolutely valid reflecting walls of the resonator. Thus the generation conditions are equal to those obtained in the previous papers [1–5], and are accurate to the numerical coefficient. It should be noted that the threshold conditions for the generation of fractional harmonics in the resonant fluid-filled cavity were observed experimentally by

Adler and Breazeale [11]; in this case the problem associated with the nonlinear effects under resonator wall vibrations was considered.

IV. SASER OPERATION IN NONLINEAR MODE

Now, let us analyze Eqs. (11a) and (11b) for all time. We will seek the useful mode in the same form as above and the concentration as

$$n(z,t)=C(t)+D(t)\cos k_L z. \tag{21}$$

Let us substitute Eqs. (12) and (21) into Eqs. (11a) and (11b), and then average Eqs. (11a) and (11b) with weight function 1, and Eq. (11b) with weight function $\cos k_L z$ over z. As a result one can obtain the set which describes a nonlinear wave propagating in the resonator, and the change of the spatial and total concentration of the bubbles:

$$\left[\frac{\omega^2}{c_l^2}-k_L^2-\frac{2i\omega}{c_l^2}\frac{d}{dt}\right]P_0=-4\pi P_0AC-4\pi P_EAD, \tag{22}$$

$$\frac{dC}{dt}=I_1+\frac{\gamma}{n_0P_E}\left[\left(C^2+\frac{D^2}{2}\right)P_E^2+\left(\frac{C^2}{2}+\frac{3}{8}D^2\right)P_0P_0^* +CDP_E(P_0+P_0^*)\right],$$

$$\frac{dD}{dt}=\frac{k_L^2}{\xi}[A^*P_0(CP_E+\tfrac{1}{4}DP_0^*)+AP_0^*(CP_E+\tfrac{1}{4}DP_0)] +\frac{2\gamma}{n_0P_E}\left[CDP_E^2+\frac{3}{4}CDP_0P_0^*+\left(\frac{C^2}{2}+\frac{3}{8}D^2\right)\times P_E(P_0+P_0^*)\right].$$

The results of the numerical solution of this set are given in Figs. 2 and 3 for the following parameters: $c_l=1500$ m/s, $\mu_l=10^{-3}$ N/m, $n_0=10^8$ m^{-3}, $\omega=2\times10^6$ Hz, $\delta_L=0.004$, $\rho_l=1000$ kg/m^3, $R_0=10^{-5}$ m, $\omega_0=2\times10^5$ Hz, $P_{\rm st}=19$ kPa, $P_E=66.5$ kPa, and $I_1=10^{-2}$ m^{-3}s^{-1}. It can be seen that the total number of emitters tends to balance between the formation and the coagulation of bubbles. The useful mode comes to saturation. It approximately equals 126 times the starting pressure.

V. EMISSION OF A SASER AT STEADY STATE

As indicated above, the state of an active medium with a spatially homogeneous concentration of bubbles becomes unstable as a sound wave passes through it. The concentration becomes periodical in space due to the action of acoustic radiation forces. Consider the bunching of bubbles only in the sense of radiation, i.e., along the z axis. The bubbles will come together at the planes where their translational velocity $\mathbf{U}$ equals zero. As Eq. (2) indicates, $\mathbf{U}\sim\sin kz$, i.e., $\mathbf{U}=0$ at planes which are perpendicular to the z axis, and spaced at intervals π/k. As is easy to see, planes with stable and unstable equilibra of the bubble will be alternated. Thus, the bubbles will be bunched at planes spaced at intervals of $2\pi/k$. As this takes place, the bubbles with radii above and below the resonance radius will be grouped in different parts of the standing wave in the resonator. The sound fields scattered from these planes are added in phase and, hence, amplified.

Let us consider a SASER with a cylindrical resonator. All bubbles are assumed to be equal. In the steady regime we consider that there are many emitters of piston type in the

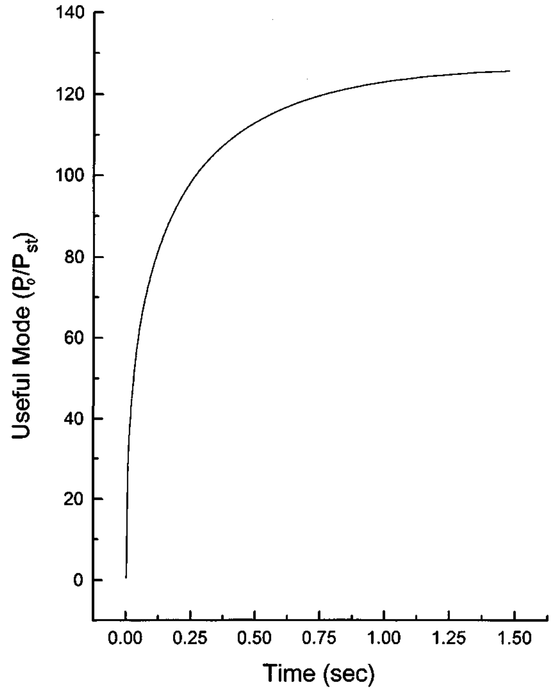

FIG. 2. The variation of the normalized useful mode with normalized time.

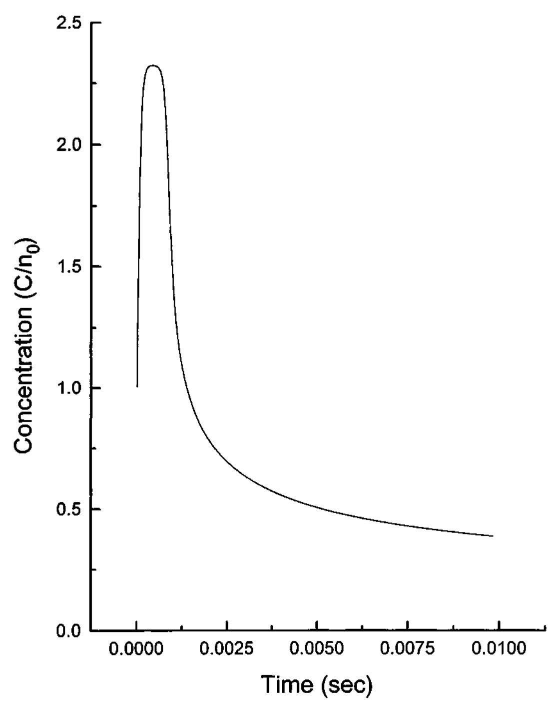

FIG. 3. The variation of the normalized bubble concentration with normalized time.

active medium which are placed $2\pi/k$ apart.

The emission of radiation by a piston emitter is described in the Fraunhofer approximation by the well-known equation [12]:

$$\phi(r,\theta) = -\frac{R^2}{r}\frac{J_1(kR\sin\theta)}{kR\sin\theta}\exp(-ikr)u_0. \tag{23}$$

Here ϕ is the velocity potential (we use cylindrical coordinates), R is the radius of the emitter, and u_0 is the amplitude of the oscillations.

In the case of a SASER the resulting potential is the sum of the potentials from each plane of bubbles. We are interested in the emission away from the resonator. Hence only phase factors exp(ikr) vary, i.e.,

$$\phi_{\mathrm{res}}(r,\theta) = -\frac{R^2}{r}\frac{J_1(kR\sin\theta)}{kR\sin\theta}\sum_{s=1}^{N}\exp(-ikr_s)u_0, \tag{24}$$

where N is the number of planes, and the quantity

$$r_s \approx r\left(1+\frac{2\pi(s-1)}{kr}\cos\theta\right) \tag{25}$$

is the distance between the sth plane and the observed point $(r_1 = r)$.

Let us consider the direction pattern for the SASER. The intensity of the emitted wave is proportional to the square of potential (24). If the number of planes N is sufficiently large (i.e., $N \approx 32$ for $\omega = 2\times 10^5$ kHz and $L = 1.5$ m) then it can easily be found that the intensity in the main direction $(\theta = 0)$ will increase and the side lobes will be suppressed.

By these means the emission of a SASER is characterized by high directionality as compared with common piston emitters. However, there is some limitation to these conclusions. At steady state the sound will undergo losses and reflections. This effects will deteriorate particle bunching.

VI. CONCLUSIONS

In the present paper we considered the operation of a SASER for all time. The coagulation of the bubbles plays no part in the evaluation of the starting pressure for a SASER. This comes as no surprise, because the initial pressure does not depend on the number of emitters at all. The larger the concentration, the larger not only the useful mode but the absorption of sound in the medium. Both these effects compete with each other.

The bubbles have a dominant role in the energy dissipation in the SASER, i.e., $\beta_0 \sim n_0$. However, β_0 contains not only the bubble term, but others as well. For example, there is a term resulting from the liquid viscosity. However, these additional contributions are negligibly small in comparison with the bubble contribution.

During the operation the SASER emission goes to saturation. This effect analogous to that in the free-electron laser theory. Estimations of the maximal value of the useful wave present readily achievable values. Thus there is a good reason to think that the suggested scheme is realistic.

ACKNOWLEDGMENT

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